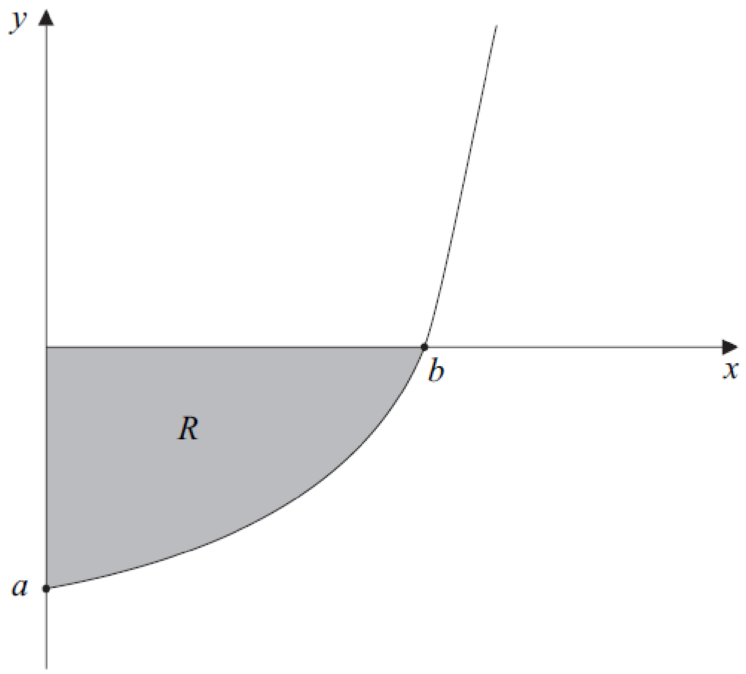
# Exponentials & Logarithms Questions

**5** The diagram shows part of the graph of $y = e^{2x} - 9$. The graph cuts the coordinate axes at $(0, a)$ and $(b, 0)$.




(a) State the value of $a$, and show that $b = \ln 3$. *(3 marks)*

(b) Show that $y^2 = e^{4x} - 18e^{2x} + 81$. *(1 mark)*

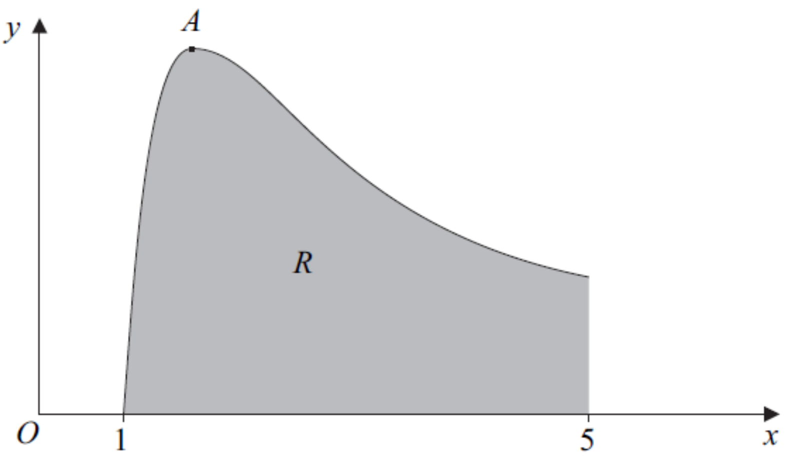
(c) The shaded region $R$ is rotated through 360° about the $x$-axis. Find the volume of the solid formed, giving your answer in the form $\pi(p \ln 3 + q)$, where $p$ and $q$ are integers. *(6 marks)*

(d) Sketch the curve with equation $y = |e^{2x} - 9|$ for $x \geqslant 0$. *(2 marks)*

---

**9** (a) Given that $y = x^{-2} \ln x$, show that $\dfrac{dy}{dx} = \dfrac{1 - 2\ln x}{x^3}$. *(4 marks)*

(c) The sketch shows the graph of $y = x^{-2} \ln x$.




(i) Using the answer to part (a), find, in terms of e, the $x$-coordinate of the stationary point $A$. *(2 marks)*

---

**5** (a) A curve has equation $y = e^{2x} - 10e^{x} + 12x$.

(i) Find $\dfrac{dy}{dx}$. *(2 marks)*

(ii) Find $\dfrac{d^2y}{dx^2}$. *(1 mark)*

(b) The points $P$ and $Q$ are the stationary points of the curve.

(i) Show that the $x$-coordinates of $P$ and $Q$ are given by the solutions of the equation

$$e^{2x} - 5e^{x} + 6 = 0$$ *(1 mark)*

(ii) By using the substitution $z = e^{x}$, or otherwise, show that the $x$-coordinates of $P$ and $Q$ are $\ln 2$ and $\ln 3$. *(3 marks)*

(iii) Find the $y$-coordinates of $P$ and $Q$, giving each of your answers in the form $m + 12 \ln n$, where $m$ and $n$ are integers. *(3 marks)*

(iv) Using the answer to part (a)(ii), determine the nature of each stationary point.

---

(b) (i) Given that $y = x \ln x$, find $\dfrac{dy}{dx}$. *(2 marks)*

(ii) Hence, or otherwise, find $\int \ln x \, dx$. *(2 marks)*

(iii) Find the exact value of $\int_1^5 \ln x \, dx$. *(2 marks)*

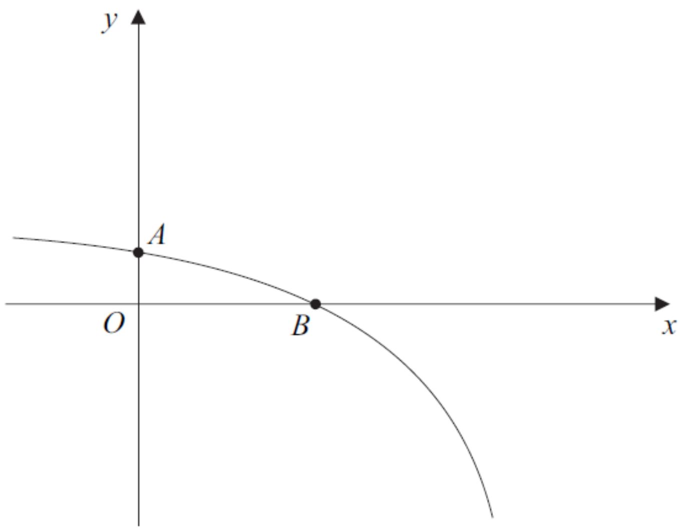
---

(b) (i) Find $\dfrac{dx}{dy}$ when $x = 2y^3 + \ln y$. *(1 mark)*

(ii) Hence find an equation of the tangent to the curve $x = 2y^3 + \ln y$ at the point $(2,1)$. *(3 marks)*

---

**9** The sketch shows the graph of $y = 4 - e^{2x}$. The curve crosses the $y$-axis at the point $A$ and the $x$-axis at the point $B$.



(a) (i) Find $\int (4 - e^{2x})dx$. *(2 marks)*

(ii) Hence show that $\int_0^{\ln 2} (4 - e^{2x})dx = 4\ln 2 - \frac{3}{2}$. *(2 marks)*

(b) (i) Write down the $y$-coordinate of $A$. *(1 mark)*

(ii) Show that $x = \ln 2$ at $B$. *(2 marks)*

(c) Find the equation of the normal to the curve $y = 4 - e^{2x}$ at the point $B$. *(4 marks)*

(d) Find the area of the region enclosed by the curve $y = 4 - e^{2x}$, the normal to the curve at $B$ and the $y$-axis. *(3 marks)*

---

**1** (a) Differentiate $\ln x$ with respect to $x$. *(1 mark)*

(b) Given that $y = (x + 1)\ln x$, find $\frac{dy}{dx}$. *(2 marks)*

(c) Find an equation of the normal to the curve $y = (x + 1)\ln x$ at the point where $x = 1$.

---

**7** (a) A curve has equation $y = (x^2 - 3)e^x$.

(i) Find $\frac{dy}{dx}$. *(2 marks)*

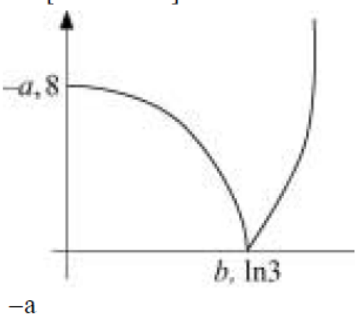
(ii) Find $\frac{d^2y}{dx^2}$. *(2 marks)*

(b) (i) Find the $x$-coordinate of each of the stationary points of the curve. *(4 marks)*

(ii) Using your answer to part (a)(ii), determine the nature of each of the stationary points. *(2 marks)*

# Exponentials & Logarithms Answers

| | | | | |
|---|---|---|---|---|
| **5(a)** | $a=-8$ | B1 | | |
| | $e^{2x}-9=0$ | M1 | | |
| | $e^{2x}=9$ | | | |
| | $2x=\ln 9$ | | | |
| | $x=\ln 3$ | A1 | 3 | AG Condone verification |
| **(b)** | $\left(e^{2x}-9\right)^2=e^{4x}-18e^{2x}+81$ | B1 | 1 | AG |
| **(c)** | $V=\pi\int y^2\,(dx)$ | B1 | | |
| | $=(\pi)\int e^{4x}-18e^{2x}+81\ dx$ | M1 | | |
| | $=(\pi)\left[\frac{e^{4x}}{4}-9e^{2x}+81x\right]_0^{\ln 3}$ | M1<br>A1 | | $1^{ST}$ or $2^{nd}$ term correct<br>All correct |

| | | | | |
|---|---|---|---|---|
| | $=(\pi)\left[\frac{e^{4x}}{4}-9e^{2x}+81x\right]_0^{\ln 3}$ | M1<br>A1 | | $1^{ST}$ or $2^{nd}$ term correct<br>All correct |
| | $=(\pi)\left[\left(\frac{e^{\ln 81}}{4}-9e^{\ln 9}+81\ln 3\right)-\left(\frac{1}{4}-9\right)\right]$ | m1 | | Attempt at limits with ln3 |
| | $=\pi[81\ln 3-52]$ | A1 | 6 | |
| **(d)** |  | M1 | | Modulus graph |
| | | A1F | 2 | All correct |
| | **Total** | | **12** | |

| | | | | |
|---|---|---|---|---|
| **9(a)** | $y=x^{-2}\ln x$ | | | |
| | $\frac{dy}{dx}=x^{-2}\frac{1}{x}-2x^{-3}\ln x$ | M1<br>A1 A1 | | Use of product or quotient<br>each term |
| | $=\frac{1-2\ln x}{x^3}$ | A1 | 4 | Convincing argument $x^{-2}\times\frac{1}{x}=x^{-3}$<br>AG |

| | | | | |
|---|---|---|---|---|
| (c)(i) | At $A$, $\frac{dy}{dx} = 0$<br>$1 - 2\ln x = 0$<br>$\ln x = \frac{1}{2}$<br>$x = e^{\frac{1}{2}}$ | M1<br><br>A1 | <br><br>2 | Attempt at $\ln x = k$ |

| | | | | |
|---|---|---|---|---|
| 5(a) | $y = e^{2x} - 10e^x + 12x$ | | | |
| (i) | $\frac{dy}{dx} = 2e^{2x} - 10e^x + 12$ | B1<br>B1 | <br>2 | $2e^{2x}$<br>remaining terms correct, no extras |
| (ii) | $\frac{d^2y}{dx^2} = 4e^{2x} - 10e^x$ | B1F | 1 | ft 1 slip |
| (b)(i) | $2e^{2x} - 10e^x + 12 = 0$<br>$e^{2x} - 5e^x + 6 = 0$ | B1 | 1 | AG (be convinced) |
| (ii) | $z^2 - 5z + 6 = 0$ | M1 | | use of $z = e^x$ oe |
| | $z = 2, 3$<br>$z = 2,\ e^x = 2$<br>$x = \ln 2$ | M1 | | finding $e^x =$ their 2,3 |
| | $z = 3,\ e^x = 3$<br>$x = \ln 3$ | A1 | 3 | all correct AG<br>SC: verification<br>$\ln 2$ (B1)<br>$\ln 3$ (B1) |
| (iii) | $x = \ln 2$ :<br>$y = e^{2\ln 2} - 10e^{\ln 2} + 12\ln 2$<br>or $2^2 - 10 \times 2 + 12\ln 2$ | M1 | | either substitution of their $x = \ln 2$ $\left(e^x = 2\right)$ or their $x = \ln 3$ $\left(e^x = 3\right)$ |
| | $= 4 - 20 + 12\ln 2$<br>$= -16 + 12\ln 2$ | A1 | | |
| | $x = \ln 3$ :<br>$y = e^{2\ln 3} - 10e^{\ln 3} + 12\ln 3$<br>$= 9 - 30 + 12\ln 3$<br>$= -21 + 12\ln 3$ | A1 | 3 | |
| (iv) | $x = \ln 2$ :<br>$\frac{d^2y}{dx^2} = 4e^{2\ln 2} - 10e^{\ln 2}$ | M1 | | use of; in either of their $e^x = 2,3$ into their $\frac{d^2y}{dx^2}$ |

| | | | | |
|---|---|---|---|---|
| | $= 16 - 20 = -4$<br>$\therefore$ maximum | A1 | | CSO |
| | $x = \ln 3$ :<br>$\frac{d^2y}{dx^2} = 4e^{2\ln 3} - 10e^{\ln 3}$<br>$= 36 - 30 = 6$<br>$\therefore$ minimum | A1 | 3 | CSO |
| | **Total** | | **13** | |

---

| | | | | |
|---|---|---|---|---|
| **(b)(i)** | $y = x \ln x$<br>$\frac{dy}{dx} = x \times \frac{1}{x} + \ln x$ | M1 | | use of product rule (only differentiating, 2 terms with + sign) |
| | $= \ln x + 1$ | A1 | 2 | |
| **(ii)** | $\int (\ln x + 1) dx = x \ln x$ | M1 | | OE; attempt at parts with $u = \ln x$ |
| | $\int \ln x \, dx = x \ln x - x (+c)$ | A1 | 2 | |
| **(iii)** | $\int_1^5 \ln x \, dx = [x \ln x - x]_1^5$<br>$= (5\ln 5 - 5) - (1\ln 1 - 1)$ | M1 | | correct substitution of limits into their (ii) provided $\ln x$ is involved |
| | $5\ln 5 - 4$ | A1 | 2 | ISW |
| | **Total** | | **9** | |

---

| | | | | |
|---|---|---|---|---|
| **(b)(i)** | $x = 2y^3 + \ln y$<br>$\frac{dx}{dy} = 6y^2 + \frac{1}{y}$ | B1 | 1 | |
| **(ii)** | At $(2,1)$<br>$\frac{dx}{dy} = 6 + 1 = 7$ | M1 | | |
| | $\frac{dy}{dx} = \frac{1}{7}$ | A1√ | | May be implied |
| | $(y-1) = \frac{1}{7}(x-2)$ | A1 | 3 | OE |

---

| | | | | |
|---|---|---|---|---|
| 9(a)(i) | $\int\left(4-e^{2x}\right)dx$ | | | |
| | $=4x-\frac{1}{2}e^{2x}\ (+c)$ | B1 | | $4x$ |
| | | B1 | 2 | $-\frac{1}{2}e^{2x}$ |
| (ii) | $\int_0^{\ln 2}=\left[4x-\frac{1}{2}e^{2x}\right]_0^{\ln 2}$ | | | |
| | $=\left[4\ln 2-\frac{1}{2}e^{2\ln 2}\right]-\left[(0)-\frac{1}{2}\left(e^0\right)\right]$ | M1 | | Substitute both ln 2 and 0 correctly into an integrated expression |
| | $=4\ln 2-2+\frac{1}{2}$ | | | Convincing |
| | $=4\ln 2-\frac{3}{2}$ | A1 | 2 | **AG** |
| (b)(i) | $x=0$ | | | |
| | $y=4-1=\underline{\underline{3}}$ | B1 | 1 | |
| (ii) | At $B$, $y=0$ | | | |
| | $4-e^{2x}=0$ | M1 | | Or reverse argument |
| | $e^{2x}=4$ | | | |
| | $x=\ln 2$ | A1 | 2 | **AG** |
| (c) | $\frac{dy}{dx}=-2e^{2x}$ | B1 | | |
| | $x=\ln 2$, Gradient $=-2e^{2\ln 2}$ | M1 | | $x=\ln 2$ into $ke^{2x}$ |
| | $=-8$ | | | |
| | Gradient normal $=\frac{1}{8}=\frac{1}{2e^{2\ln 2}}$ | A1 | | OE |
| | Equation $y=\frac{1}{8}x-\frac{1}{8}\ln 2$ | A1 | 4 | OE |
| (d) | When $x=0$ | | | |
| | $y=-\frac{1}{8}\ln 2$ | M1 | | Attempt to integrate their line and substitute $x=0, \ln 2$ |
| | Area $\Delta$ $=\frac{1}{16}(\ln 2)^2$ condone – ve sign | A1√ | | $\frac{1}{2}(\text{their } y)\times\ln 2$ |
| | $=0.03$ | | | |
| | Total area $=4\ln 2-\frac{3}{2}+\frac{1}{16}(\ln 2)^2=1.30$ | A1 | 3 | CSO |
| | AWRT | | | |
| | **Total** | | **14** | |

| | | | | |
|---|---|---|---|---|
| 1(a) | $y = \ln x$ <br> $\frac{dy}{dx} = \frac{1}{x}$ | B1 | 1 | penalise + $c$ once on 1(a) or 2(a) |
| (b) | $y = (x+1)\ln x$ <br> $\frac{dy}{dx} = (x+1) \times \frac{1}{x} + \ln x$ | M1 <br> A1 | 2 | product rule |
| (c) | $y = (x+1)\ln x$ <br> $\frac{dy}{dx} = \frac{1}{x} + 1 + \ln x$ | | | |
| | $x = 1$: $\frac{dy}{dx} = 1 + 1 = 2$ | M1 | | substitute $x = 1$ into their $\frac{dy}{dx}$ |
| | Grad normal $= -\frac{1}{2}$ | M1 <br> A1 | | use of $m_1 m_2 = -1$ <br> CSO |
| | $y = -\frac{1}{2}(x-1)$ | A1 | 4 | OE |
| | **Total** | | **7** | |

---

| | | | | |
|---|---|---|---|---|
| 7(a)(i) | $y = (x^2 - 3)e^x$ <br> $\frac{dy}{dx} = (x^2 - 3)e^x + 2xe^x$ | M1 <br> A1 | 2 | product rule |
| (ii) | $\frac{d^2y}{dx^2} = (x^2 - 3)e^x + 2xe^x + 2xe^x + 2e^x$ | M1 <br> A1 | 2 | product rule from their $\frac{dy}{dx}$ |
| (b)(i) | $\frac{dy}{dx} = 0$ <br> $\Rightarrow e^x(x^2 + 2x - 3) = 0$ | M1 | | $e^x f(x) = 0$ from $\frac{dy}{dx} = 0$ |
| | $e^x(x+3)(x-1) = 0$ | m1 | | attempt at factorising or use of formula |
| | $\therefore x = -3, 1$ | A1 <br> A1 | 4 | first correct solution <br> second correct solution, and no others <br> SC No working shown: <br> $x = -3$ B2, $x = 1$ B2 |
| (ii) | $x = -3$ $y'' = -4e^x$ max $(-0.2)$ | M1 | | Condone slip |
| | $x = 1$ $y'' = 4e^x$ min $(10.9)$ | A1 | 2 | |
| | **Total** | | **10** | |