Probability Generating Functions

Questions

Q1.

The discrete random variable *X* has probability generating function

$$G_X(t) = k\ln\left(\frac{2}{2-t}\right)$$

where *k* is a constant.

(a) Find the exact value of *k*

(1)

(b) Find the exact value of Var(*X*)

(7)

(c) Find P(*X* = 3)

(4)

(Total for question = 12 marks)

Q2.

A discrete random variable *X* has probability generating function given by

$$G_X(t) = \frac{1}{64}(a + bt^2)^2$$

where *a* and *b* are positive constants.

(a) Write down the value of P(*X* = 3)

(1)

Given that P(*X* = 4) = $\frac{25}{64}$

(b) (i) find P(*X* = 2)

(7)

(ii) find E(*X*)

(3)

The random variable *Y* = 3*X* + 2

(c) Find the probability generating function of *Y*

(2)

(Total for question = 13 marks)

Q3.

The probability generating function of the random variable X is

$$G_X(t) = k(1 + 2t)^5$$

where k is a constant.

(a) Show that $k = \dfrac{1}{243}$

(2)

(b) Find $P(X = 2)$

(2)

(c) Find the probability generating function of $W = 2X + 3$

(2)

The probability generating function of the random variable Y is

$$G_Y(t) = \frac{t(1+2t)^2}{9}$$

Given that X and Y are independent,

(d) find the probability generating function of $U = X + Y$ in its simplest form.

(2)

(e) Use calculus to find the value of $\text{Var}(U)$

(6)

(Total for question = 14 marks)

Q4.

The probability generating function of the discrete random variable X is given by

$$G_X(t) = k(3 + t + 2t^2)^2$$

(a) Show that $k = \dfrac{1}{36}$

(2)

(b) Find $P(X = 3)$

(2)

(c) Show that $\text{Var}(X) = \dfrac{29}{18}$

(8)

(d) Find the probability generating function of $2X + 1$

(2)

(Total for question = 14 marks)

Mark Scheme – Probability Generating Functions

Q1.

Qu	Scheme	Marks	AO
(a)	$G(1)=1 \Rightarrow k\ln 2=1$ so $k=\dfrac{1}{\ln 2}$	B1	2.1
		(1)	
(b)	$\left\{G(t)=\dfrac{1}{\ln 2}[\ln 2-\ln(2-t)]\right\} \Rightarrow G'(t)=\dfrac{1}{\ln 2}\left[\dfrac{1}{2-t}\right]$ or $\dfrac{1}{\ln 2}(2-t)^{-1}$	M1 A1	2.1 1.1b
	$[E(X)=]\ G'(1)=\dfrac{1}{\ln 2}$	A1	1.1b
	$G''(t)=\dfrac{1}{\ln 2}\times\left[\dfrac{1}{(2-t)^2}\right]$	M1 A1	2.1 1.1b
	$\text{Var}(X)=G''(1)+G'(1)-[G'(1)]^2=\dfrac{1}{\ln 2}+\dfrac{1}{\ln 2}-\left(\dfrac{1}{\ln 2}\right)^2$	M1	2.1
	$=\dfrac{1}{\ln 2}\left(2-\dfrac{1}{\ln 2}\right)$	A1	1.1b
		(7)	
(c)	$P(X=3)=$ coefficient of t^3 by Maclaurin need $G'''(0)$	M1	3.1a
	$G'''(t)=\dfrac{1}{\ln 2}\dfrac{2}{(2-t)^3}$	A1ft	1.1b
	$P(X=3)=\dfrac{G'''(0)}{3!}$	M1	3.2a
	$=\dfrac{\frac{1}{4\ln 2}}{6}=\dfrac{1}{24\ln 2}=0.0601122\ldots$ awrt **0.0601**	A1	1.1b
		(4)	
		(12 marks)	

	Notes
(a)	B1 for finding k (must be exact)
(b)	1st M1 for an attempt to differentiate $G(t)$ e.g. $A(2-t)^{-1}$ (o.e.) 1st A1 for a correct first derivative (condone k or use of $\frac{1}{\ln 2}=$ awrt 1.44) 2nd A1 for correct $E(X)$ or $G'(1)$ (allow awrt 1.44 calc: 1.442695…but not k) seen anywhere 2nd M1 for attempting second derivative (ft their $G'(t)$) 3rd A1 for a correct 2nd derivative (condone k or use of $\frac{1}{\ln 2}=$ awrt 1.44) 3rd M1 for a correct method for $\text{Var}(X)$ (some substitution into the correct formula) 4th A1 for $\dfrac{1}{\ln 2}\left(2-\dfrac{1}{\ln 2}\right)$ o.e. but must simplify i.e. collect like terms [Mark final answer – penalise incorrect log work etc] NB 0.8040211.. is A0 unless exact answer seen
(c)	1st M1 for a suitable strategy to solve the problem (finding link with Maclaurin) Need mention of coefficient of t^3 **and** $[G'''(t)$ or $G'''(0)]$(condone $G'''(1)$) 1st A1ft for 3rd derivative , ft their 2nd derivative in (b) (provided $G''(t)$ not const) Correct $G'''(t)$ or $G'''(0)$ scores 1st M1 1st A1ft 2nd M1 for translating Maclaurin to probability (a correct expression) 2nd A1 for $\frac{1}{24\ln 2}$ or awrt 0.0601
ALT	**Log series** 1st M1 attempt to write $G(t)$ in suitable form as far as: $k[\ln 2-\ln(2[1-\frac{t}{2}])]$ 1st A1 reaching $-k\ln(1-\frac{t}{2})$ 2nd M1 use of $-\ln(1-x)$ series (some correct substitution) NB $G(t)=\frac{1}{\ln 2}\left(\frac{t}{2}+\frac{t^2}{8}+\frac{t^3}{24}+\ldots\right)$

Q2.

Question	Scheme	Marks	AOs
(a)	$P(X=3) = \underline{0}$	B1	1.1b
		(1)	
(b)(i)	Coefficient of $t^4 = \frac{1}{64}b^2$	M1	2.1
	$\frac{1}{64}b^2 = \frac{25}{64}$	M1	1.1b
	$b = 5$ (reject $b = -5$ since $b > 0$)	A1	2.3
	$G_X(1) = 1$ $\frac{1}{64}(a + "5")^2 = 1$	M1	2.1
	$a = 3$ (reject $a = -13$ since $a > 0$)	A1	1.1b
	$P(X=2) =$ coefficient of $t^2 = \frac{1}{64}(2ab)$	M1	3.4
	$= \underline{\frac{15}{32}}$	A1	1.1b
		(7)	
(ii)	$E(X) = G'_X(1)$	M1	2.1
	$G'_X(t) = \frac{2}{64}("3"+"5"t^2)\times"10"t$ or $G'_X(t) = \frac{1}{64}("60"t + "100"t^3)$	M1	1.1b
	$G'_X(1) = 2.5$	A1ft	1.1b
		(3)	
(c)	$G_Y(t) = t^2 G_X(t^3)[= \frac{t^2}{64}(a + b(t^3)^2)^2]$	M1	3.1a
	$G_Y(t) = \frac{t^2}{64}("3"+"5"t^6)^2$	A1ft	1.1b
		(2)	
			(13 marks)

Notes	
(a)	**B1:** 0 (Since there is no term in t^3)
(b)(i)	**M1:** Realising that $\frac{1}{64}b^2$, the coefficient of t^4, is needed **M1:** Equating their coefficient of t^4 to $\frac{25}{64}$ with an attempt to find b **A1:** $b = 5$ only **M1:** Realising that $G_X(1) = 1$ is required **A1:** $a = 3$ only **M1:** Finding coefficient of t^2 with their $a > 0$ and $b > 0$ **A1:** $\frac{15}{32}$ (condone awrt 0.469)
(b)(ii)	**M1:** Realising $G'_X(1)$ is needed **M1:** Attempt to differentiate $G_X(t)$ with their values of a and b **A1ft:** 2.5 (ft (3sf) their values of a and b, $a > 0$ and $b > 0$) $E(X) = \frac{ab+b^2}{16}$ Alternative: **M1:** Realising $X = 0$, 2 and 4 only **M1:** $[0 \times P(X=0)] + 2\times P(X=2) + 4\times P(X=4)$
(c)	**M1:** either $G_X(t^3)$ or $\times t^2$ or using $Y = 2, 8, 14$ **A1ft:** ft their values of a and b, $a > 0$ and $b > 0$ $G_Y(t) = \frac{t^2}{64}("3"+"5"t^6)^2$ or $G_Y(t) = \frac{t^2}{64}("9"+"30"t^6 + "25"t^{12})$ or $G_Y(t) = \frac{1}{64}("9t^2"+"30"t^8 + "25"t^{14})$

Q3.

Question	Scheme	Marks	AOs
(a)	$\mathrm{G}_X(1) = 1$	M1	2.1
	$k \times 3^5 = 1 \quad \therefore k = \dfrac{1}{243}$ *	A1*cso (2)	1.1b
(b)	$\mathrm{P}(X=2)$ is coefficient of t^2 so $\mathrm{G}_X(t) = k\left(... + {}^5C_2(2t)^2 + ...\right)$	M1	1.1b
	$\mathrm{P}(X=2) = \dfrac{40}{243}$	A1 (2)	1.1b
(c)	$\mathrm{G}_W(t) = \dfrac{t^3}{243}\left(1+2(t^2)\right)^5$	M1	3.1a
	$\mathrm{G}_W(t) = \dfrac{t^3}{243}(1+2t^2)^5$	A1 (2)	1.1b
(d)	$G_U(t) \ = \dfrac{1}{243}(1+2t)^5 \times \dfrac{t(1+2t)^2}{9}$	M1	3.1a
	$= \dfrac{t(1+2t)^7}{2187}$	A1 (2)	1.1b
(e)	$\mathrm{G}_U'(t) = \dfrac{14t(1+2t)^6}{2187} + \dfrac{(1+2t)^7}{2187}$	M1	2.1
	$\mathrm{G}_U'(1) = \dfrac{17}{3}$	A1ft	1.1b
	$\mathrm{G}_U''(t) = \dfrac{168t(1+2t)^5}{2187} + \dfrac{14(1+2t)^6}{2187} + \dfrac{14(1+2t)^6}{2187}$	M1	2.1
	$\mathrm{G}_U''(1) = 28$	A1	1.1b
	$\mathrm{Var}(U) = "28" + "\dfrac{17}{3}" - \left("\dfrac{17}{3}"\right)^2$	M1	2.1
	$= \dfrac{14}{9}$	A1 (6)	1.1b
ALT(e)	$\mathrm{G}_X''(t) = A(1+2t)^3$	M1	
	$\mathrm{G}_X'(1) = \dfrac{10}{3}$ and $\mathrm{G}_X''(1) = \dfrac{80}{9}$	A1ft	
	$\mathrm{G}_Y''(t) = H(8+24t)$	M1	
	$\mathrm{G}_Y'(1) = \dfrac{7}{3}$ and $\mathrm{G}_Y''(1) = \dfrac{32}{9}$	A1	
	Using $\mathrm{G}_U''(1) + \mathrm{G}_U'(1) - \left(\mathrm{G}_U'(1)\right)^2$ to find $\mathrm{Var}(X)$, Var Y and Var U	M1	
	$\dfrac{14}{9}$ or awrt1.56	A1	
		(14 marks)	

Notes:		
(a)	M1:	Stating $G_X(1) = 1$ eg $G_X(1) = k(1+2)^5 = 1$ $k(1+2)^5 = 1$ Allow Verification $\frac{1}{243}\times 3^5 = 1$
	A1*:	Fully correct proof with no errors Substituting $t=1$ Verification need therefore $G_X(1) = 1$
(b)	M1:	Attempting to find the coefficient of t^2
	A1:	$\frac{40}{243}$ or awrt 0.165
(c)	M1:	Realising the need to multiply through by t^3 or subst t^2 for t
	A1:	$\frac{t^3}{243}\left(1+2t^2\right)^5$ oe eg $\frac{t^3}{243}\left(1+10t^2+40t^4+80t^6+80t^8+32t^{10}\right)$
(d)	M1:	Realising the need to use $G_U(t) = G_X(t)\times G_Y(t)$
	A1:	$\frac{t(1+2t)^7}{2187}$ oe
(e)	M1:	For an attempt to differentiate G (u) e.g $G_U'(t) = At(1+2t)^6 + B(1+2t)^7$ ft their part(d) if in the form $kt(1+2t)^n$ where $n\geqslant 5$
	A1ft:	$\frac{17}{3}$ or awrt 5.67
	M1:	For attempting second derivative eg $G_U''(t) = Ct(1+2t)^5 + D(1+2t)^6$ ft their part(d) if in the form $kt(1+2t)^n$ where $n\geqslant 5$
	A1	28
	M1:	Using $G_U''(1) + G_U'(1) - \left(G_U'(1)\right)^2$ ft their values
	A1:	$\frac{14}{9}$ or awrt1.56

Q4.

Question	Scheme	Marks	AOs
(a)	$G_X(1)=1$ gives	M1	2.1
	$k\times 6^2=1$ so $k=\frac{1}{36}$ *	A1*cso	1.1b
		(2)	
(b)	$P(X=3)=$ coefficient of t^3 so $G_X(t)=k(\ldots+4t^3\ldots)$	M1	1.1b
	$[\,P(X=3)=]\ \underline{\frac{1}{9}}$	A1	1.1b
		(2)	
(c)	$G'_X(t)=2k(3+t+2t^2)\times(1+4t)$	M1	2.1
	$E(X)=G'_X(1)=2k(3+1+2)\times(1+4)$	M1	1.1b
	$=\underline{\frac{5}{3}}$	A1	1.1b
	$G''_X(t)=2k\left[(3+t+2t^2)\times 4+(1+4t)^2\right]$	M1 A1	2.1 1.1b
	$G''_X(1)=2k[6\times4+5^2]$ $\left\{=\frac{49}{18}\right\}$	M1	1.1b
	$\text{Var}(X)=G''_X(1)+G'_X(1)-[G'_X(1)]^2=\frac{49}{18}+\frac{5}{3}-\frac{25}{9}$	M1	2.1
	$=\underline{\frac{29}{18}}$ *	A1*cso	1.1b
		(8)	
(d)	$G_{2X+1}(t)=\frac{t}{36}\left(3+t^2+2(t^2)^2\right)^2$ [$\times t$ or sub t^2 for t]	M1	3.1a
	$=G_{2X+1}(t)=\frac{t}{36}(3+t^2+2t^4)^2$	A1	1.1b
		(2)	
			(14 marks)

Notes	
(a)	**M1:** Stating $G_X(1)=1$ **A1*cso:** Fully correct proof with no errors
(b)	**M1:** Attempting to find the coefficient of t^3. May be implied by obtaining $\frac{1}{9}$ or awrt 0.11 **A1:** $\frac{1}{9}$, allow awrt 0.111

	Notes (continued)
(c)	**M1**: Attempting to find $G'_X(t)$. Allow Chain rule or multiplying out the brackets and differentiating **M1**: Substituting $t=1$ into $G'_X(t)$ **A1**: $\frac{5}{3}$, allow awrt 1.67 **M1**: Attempting to find $G''_X(t)$ **A1**: $2k\left[\left(3+t+2t^2\right)\times 4+\left(1+4t\right)^2\right]$ or $k(48t^2+24t+26)$ o.e. **A1**: $2k[6\times 4+5^2]$ o.e. **M1**: Using $G''_X(1)+G'_X(1)-\left[G'_X(1)\right]^2$ to find the Variance **A1*cso**: $\frac{29}{18}$
(d)	**M1**:Realising the need to $\times t$ or sub t^2 for t **A1**: $\frac{t}{36}\left(3+t^2+2t^4\right)^2$, or $\frac{t}{36}\left(9+6t^2+13t^4+4t^6+4t^8\right)$ o.e.