Methods in Differential Equations

Questions

Q1.

(a) Find, in the form $y = f(x)$, the general solution of the equation

$$\cos x\frac{dy}{dx} + y\sin x = 2\cos^3 x\sin x + 1, \quad 0 < x < \frac{\pi}{2}$$

(8)

Given that $y = 5\sqrt{2}$ when $x = \frac{\pi}{4}$

(b) find the value of y when $x = \frac{\pi}{6}$, giving your answer in the form $a + b\sqrt{3}$, where a and b are rational numbers to be found.

(3)

(Total for question = 11 marks)

Q2.

(a) Determine the general solution of the differential equation

$\cos x\frac{dy}{dx} + y\sin x = e^{2x}\cos^2 x$

giving your answer in the form $y = f(x)$

(3)

Given that $y = 3$ when $x = 0$

(b) determine the smallest positive value of x for which $y = 0$

(3)

(Total for question = 6 marks)

Q3.

(a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 26\sin 3x$$

(8)

(b) Find the particular solution of this differential equation for which $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

(5)

(Total for question = 13 marks)

Mark Scheme – Methods in Differential Equations

Q1.

Question Number	Scheme	Notes	Marks
	$\cos x\dfrac{dy}{dx}+y\sin x=2\cos^3 x\sin x+1$		
(a)	$\dfrac{dy}{dx}+y\tan x=2\cos^2 x\sin x+\dfrac{1}{\cos x}$	Divides by $\cos x$ LHS both terms divided RHS min 1 term divided	M1
	$I=e^{\int \tan x dx}=e^{\ln\sec x}=\sec x$	M1: Attempt integrating factor $e^{\int \tan x dx}$ needed A1: Correct integrating factor, $\sec x$ or $\dfrac{1}{\cos x}$	dM1A1
	$y\sec x=\int(2\sin x\cos x+\sec^2 x)dx$	Multiply through by their IF and integrate LHS (integration may be done later) $yI=\int(\text{their RHS})I\,dx$	M1
	$y\sec x=-\dfrac{1}{2}\cos 2x+\tan x(+c)$	M1: Attempt integration of at least one term on RHS (provided both sides have been multiplied by their IF.) OR $\sec^2 x\to K\tan x$ A1: $-\dfrac{1}{2}\cos 2x$ or equivalent integration of $2\sin x\cos x$ ($\sin^2 x$ or $-\cos^2 x$) A1: $\tan x$ constant not needed.	M1A1A1
	$y=\left(-\dfrac{1}{2}\cos 2x+\tan x+c\right)\cos x$ $y=(-\cos^2 x+\tan x+c)\cos x$ $y=(\sin^2 x+\tan x+c)\cos x$	Include the constant and deal with it correctly. Must start $y=...$ Or equivalent eg $y=-\dfrac{1}{2}\cos 2x\cos x+\sin x+c\cos x$ Follow through from the line above	A1ft
			(8)
(b)	$x=\dfrac{\pi}{4}\Rightarrow 5\sqrt{2}=.....\Rightarrow c=.....$	Substitutes for x and y and solves for c (If substitution not shown award for at least one term evaluated correctly.)	M1
	$x=\dfrac{\pi}{6}\Rightarrow y=.........$	Substitutes $x=\dfrac{\pi}{6}$ to find a value for y	M1
	$y=\dfrac{1}{2}+\dfrac{35}{8}\sqrt{3}$ or $y=0.5+4.375\sqrt{3}$	Must be in given form. Equivalent fractions allowed. ...	A1cao
			(3)
NB	(b) There may be no working shown due to use of calculator. In such cases: Final answer correct (and in required form with no decimals instead of $\sqrt{3}$ seen), score 3/3. Final answer incorrect (or decimals instead of $\sqrt{3}$ seen), score 0/3. This applies whether (a) is correct or not.		
			Total 11

Q2.

Question	Scheme	Marks	AOs
(a)	$\frac{dy}{dx} + y\tan x = e^{2x}\cos x$ $\text{IF} = e^{\int \tan x\, dx} = e^{\ln \sec x} = \sec x \Rightarrow \sec x\frac{dy}{dx} + y\sec x\tan x = e^{2x}$ $\Rightarrow y\sec x = \int e^{2x}\ dx$	M1	3.1a
	$y\sec x = \frac{1}{2}e^{2x}(+c)$	A1	1.1b
	$y = \left(\frac{1}{2}e^{2x} + c\right)\cos x$	A1	1.1b
		(3)	
(b)	$x = 0,\ y = 3 \Rightarrow c = \ldots\{2.5\}$	M1	3.1a
	$y = \left(\frac{1}{2}e^{2x} + \frac{5}{2}\right)\cos x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \ldots$	M1	1.1b
	$x = \frac{\pi}{2}$	A1	1.1b
		(3)	
		(6 marks)	

Notes:

(a)

M1: Finds the integrating factor and attempts the solution of the differential equation.

Look for I.F. $= e^{\int \tan x dx} \Rightarrow y \times$ 'their I.F.' $= \int e^{2x}\cos x \times$ 'their I.F.' dx

A1: Correct solution condone missing $+ c$

A1: Correct general solution, Accept equivalents of the form $y = f(x)$, such as $y = \frac{e^{2x}}{2\sec x} + \frac{c}{\sec x}$

(b)

M1: Uses $x = 0$ $\ y = 3$ to find the constant of integration. Allow if done as part of part (a) and allow for their answer to (a) as long as it has a constant of integration to find.

M1: Sets $y = 0$ in an equation of the form $y = (Ae^{2x} + c)\cos x$ (oe) where A is 1, 2 or $\frac{1}{2}$, with their c or constant c and makes a valid attempt to solve the equation to find a value for x. (Allow even if the constant of integration has not been found).

A1: Depends on both M's. Awrt 1.57 or $\frac{\pi}{2}$ only. There must have been an attempt to find the constant of integration, but allow from a correct answer to (a) as long as a positive value for c has been found (can be scored from implicit form).

Q3.

Question Number	Scheme	Notes	Marks
	$\frac{d^2y}{dx^2}-2\frac{dy}{dx}=26\sin 3x$		
(a)	$m^2-2m=0 \Rightarrow m=0,\ 2$	Solves AE	M1
	$(\text{CF or } y=)\,A+Be^{2x}$ or Ae^0+Be^{2x} oe	Correct CF (CF or $y=$ not needed)	A1
	$(\text{PI or } y=)\ a\cos 3x+b\sin 3x$	Correct form for PI (PI or $y=$ not needed)	B1
	$\frac{dy}{dx}=-3a\sin 3x+3b\cos 3x,\ \frac{d^2y}{dx^2}=-9a\cos 3x-9b\sin 3x$		M1A1
	M1: Differentiates twice; change of trig functions needed, ± 1 or ± 3 for coeffs for first derivative, ± 1, ± 3 or ± 9 for second derivative (1/3 etc indicates integration) A1: Correct derivatives		
	$-9a\cos 3x-9b\sin 3x+6a\sin 3x-6b\cos 3x=26\sin 3x$		
	$\therefore -9a-6b=0,\ \ -9b+6a=26 \Rightarrow a=...,b=...$	Substitutes and forms simultaneous equations (by equating coeffs) and attempts to solve for a and b Depends on the second M mark	dM1
	$a=\frac{4}{3},\ b=-2$	Correct a and b	A1
	$y=A+Be^{2x}+\frac{4}{3}\cos 3x-2\sin 3x$	Forms the GS (ft their CF and PI) Must start $y=$	A1ft **(8)**
(b)	$0=A+B+\frac{4}{3}$	Substitutes $x=0$ and $y=0$ into their GS	M1
	$\left(\frac{dy}{dx}\right)=2Be^{2x}-4\sin 3x-6\cos 3x \Rightarrow 0=2B-6$ Differentiates and substitutes $x=0$ and $y'=0$ (change of trig functions needed, ± 1 or ± 3 for coeffs)		M1
	$0=A+B+\frac{4}{3},\ 0=2B-6 \Rightarrow A=...,B=..$	Solves simultaneously to obtain values for A and B Depends on the second M mark	dM1
	$A=\frac{-13}{3},\quad B=3$	Correct values	A1
	$y=3e^{2x}-\frac{13}{3}+\frac{4}{3}\cos 3x-2\sin 3x$	Follow through their GS and A and B Must start $y=$ ...	A1ft **(5)**
			Total 13

ALT for (a)	$\frac{d^2y}{dx^2}-2\frac{dy}{dx}=26\sin 3x \Rightarrow \frac{dy}{dx}-2y=-\frac{26}{3}\cos 3x+c$	M1: Integrates both sides wrt x A1: Correct expression	M1A1
	$I=e^{\int -2dx}=e^{-2x}$	Correct integrating factor	B1
	$ye^{-2x}=\int e^{-2x}\left(-\frac{26}{3}\cos 3x+c\right)dx$	M1: Uses $yI=\int I\left(-\frac{26}{3}\cos 3x+c\right)dx$ A1: Correct expression	M1A1
	$=\frac{4}{3}e^{-2x}\cos 3x-2e^{-2x}\sin 3x-\frac{1}{2}ce^{-2x}+B$	M1: Integration by parts twice A1: Correct expression	M1A1
	$y=-\frac{1}{2}c+Be^{2x}+\frac{4}{3}\cos 3x-2\sin 3x$	Must start $y=$ ...	