The *t* - Formulae

Questions

Q1.

(a) Write down the *t*-formula for sin *x*.

(1)

(b) Use the answer to part (a)

(i) to find the exact value of sin *x* when

$$\tan\left(\frac{x}{2}\right) = \sqrt{2}$$

(ii) to show that

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

(4)

(c) Use the *t*-formulae to solve for $0 < \theta \leq 360°$

$$7\sin\theta + 9\cos\theta + 3 = 0$$

giving your answers to one decimal place.

(4)

(Total for question = 9 marks)

Q2.

(i) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to prove that

$$\cot x + \tan\left(\frac{x}{2}\right) = \operatorname{cosec} x \qquad x \neq n\pi,\ n \in \mathbb{Z}$$

(2)

(ii)

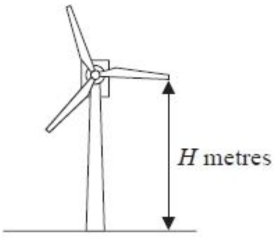

Figure 1

An engineer models the vertical height above the ground of the tip of one blade of a wind turbine, shown in Figure 1. The ground is assumed to be horizontal.
The vertical height of the tip of the blade above the ground, *H* metres, at time *x* seconds after the wind turbine has reached its constant operating speed, is modelled by the equation

$$H = 90 - 30\cos(120x)° - 40\sin(120x)° \qquad \text{(I)}$$

(a) Show that $H = 60$ when $x = 0$

(1)

Using the substitution $t = \tan(60x)°$
(b) show that equation (I) can be rewritten as

$$H = \frac{120t^2 - 80t + 60}{1 + t^2}$$

(3)

Hence find, according to the model, the value of *x* when the tip of the blade is 100 m above
the ground for the first time after the wind turbine has reached its constant operating speed.

(5)

(Total for question = 11 marks)

Q3.

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\sec x - \tan x \equiv \frac{1-t}{1+t} \qquad x \neq (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$$

(3)

(b) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ and the answer to part (a) to prove that

$$\frac{1-\sin x}{1+\sin x} \equiv (\sec x - \tan x)^2 \qquad x \neq (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$$

(3)

(Total for question = 6 marks)

Q4.

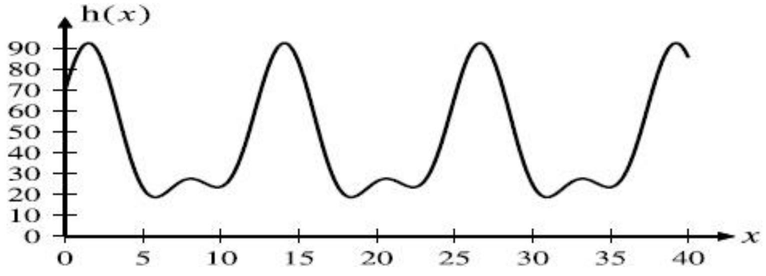

Figure 1

Figure 1 shows the graph of the function h(x) with equation

$$h(x) = 45 + 15\sin x + 21\sin\left(\frac{x}{2}\right) + 25\cos\left(\frac{x}{2}\right) \qquad x \in [0, 40]$$

(a) Show that

$$\frac{\mathrm{dh}}{\mathrm{d}x} = \frac{\left(t^2 - 6t - 17\right)\left(9t^2 + 4t - 3\right)}{2\left(1 + t^2\right)^2}$$

where $t = \tan\left(\frac{x}{4}\right)$.

(6)

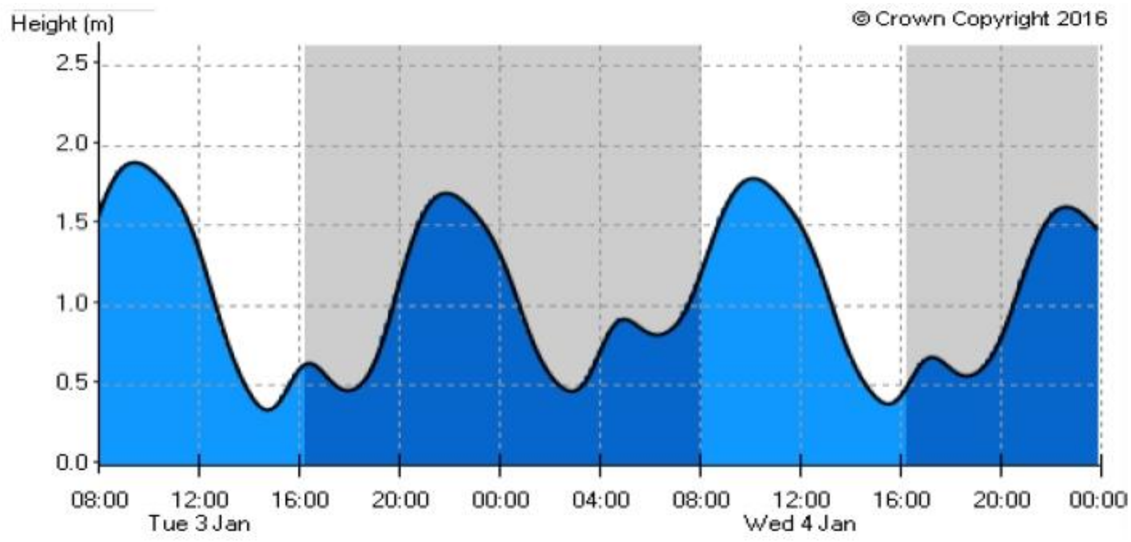

Figure 2

Figure 2 shows a graph of predicted tide heights, in metres, for Portland harbour from 08:00 on the 3rd January 2017 to the end of the 4th January 2017[1].

The graph of k h(x), where k is a constant and x is the number of hours after 08:00 on 3rd of January, can be used to model the predicted tide heights, in metres, for this period of time.

(b) (i) Suggest a value of k that could be used for the graph of k h(x) to form a suitable model.

(ii) Why may such a model be suitable to predict the times when the tide heights are at their peaks, but not to predict the heights of these peaks?

(3)

(c) Use Figure 2 and the result of part (a) to estimate, to the nearest minute, the time of the highest tide height on the 4th January 2017.

(6)

[1]Data taken on 29th December 2016 from http://www.ukho.gov.uk/easytide/EasyTide

(Total for question = 15 marks)

Mark Scheme – The *t* - Formulae

Q1.

Question	Scheme	Marks	AOs
(a)	$\{\sin x =\}\ \dfrac{2t}{1+t^2}$	B1	1.2
		(1)	
(b)(i)	$\left\{\tan\left(\dfrac{x}{2}\right)=\sqrt{2} \Rightarrow t=\sqrt{2} \Rightarrow\right\}\ \sin x = \dfrac{2(\sqrt{2})}{1+(\sqrt{2})^2}$ or $\dfrac{2(\sqrt{2})}{1+2}$	M1	1.1b
	$\sin x = \dfrac{2}{3}\sqrt{2}$ or $\dfrac{1}{3}\sqrt{8}$ or $\sqrt{\dfrac{8}{9}}$	A1	1.1b
		(2)	
(ii) Way 1	$\left\{\cos x \equiv \dfrac{\sin x}{\tan x} \Rightarrow\right\}\ \cos x = \dfrac{\frac{2t}{1+t^2}}{\frac{2t}{1-t^2}}\ ;\ = \dfrac{1-t^2}{1+t^2}$ * cso	M1;	1.1b
		A1*	2.1
		(2)	
(ii) Way 2	$\left\{\tan x \equiv \dfrac{\sin x}{\cos x} \Rightarrow\right\}\ \dfrac{2t}{1-t^2} = \dfrac{\frac{2t}{1+t^2}}{\cos x}\ ;\ \Rightarrow \cos x = \dfrac{1-t^2}{1+t^2}$ * cso	M1;	1.1b
		A1*	2.1
		(2)	
(ii) Way 3	$\{\sin^2 x + \cos^2 x \equiv 1 \Rightarrow\}\ \left(\dfrac{2t}{1+t^2}\right)^2 + \cos^2 x = 1$	M1	1.1b
	$\cos^2 x = 1 - \left(\dfrac{2t}{1+t^2}\right)^2 = \dfrac{(1+t^2)^2 - 4t^2}{(1+t^2)^2} = \dfrac{1-2t^2+t^4}{(1+t^2)^2} = \dfrac{(1-t^2)^2}{(1+t^2)^2}$ $\Rightarrow \cos x = \dfrac{1-t^2}{1+t^2}$ * cso	A1	2.1
		(2)	
(ii) Way 4	$\{o^2 + a^2 = h^2 \Rightarrow\}\ (2t)^2 + a^2 = (1+t^2)^2$	M1	1.1b
	$a^2 = (1+t^2)^2 - (2t)^2 = 1 - 2t^2 + t^4 = (1-t^2)^2$ $a = 1-t^2 \Rightarrow \cos x = \dfrac{1-t^2}{1+t^2}$ * cso	A1	2.1
		(2)	
(c)	$\{7\sin\theta + 9\cos\theta + 3 =\}\ 7\left(\dfrac{2t}{1+t^2}\right) + 9\left(\dfrac{1-t^2}{1+t^2}\right) + 3$	M1	1.1b
	$7\left(\dfrac{2t}{1+t^2}\right) + 9\left(\dfrac{1-t^2}{1+t^2}\right) + 3 = 0 \Rightarrow 14t + 9 - 9t^2 + 3 + 3t^2 = 0$ $\Rightarrow 6t^2 - 14t - 12 = 0 \Rightarrow 3t^2 - 7t - 6 = 0 \Rightarrow (t-3)(3t+2) = 0 \Rightarrow t = \ldots$	M1	1.1b
	Either $\left\{t = 3 \Rightarrow \dfrac{\theta}{2} = \arctan(3) \Rightarrow\right\}\ \theta = 2\arctan(3)$ **or** $\left\{t = -\dfrac{2}{3} \Rightarrow \dfrac{\theta}{2} = 180° + \arctan\left(-\dfrac{2}{3}\right) \Rightarrow\right\}\ \theta = 2\left(180° + \arctan\left(-\dfrac{2}{3}\right)\right)$	M1	1.1b
	$\dfrac{\theta}{2} = \{71.5650\ldots, 146.3099\ldots\} \Rightarrow \theta = \{143.1301\ldots, 292.6198\ldots\}$		
	$\theta = 143.1°, 292.6°$ (1dp)	A1	1.1b
		(4)	
			(9 marks)

Notes for Question 1	
(a)	
B1:	See scheme
(b)(i)	
M1:	Complete substitution of $t=\sqrt{2}$ into their expression from part (a)
A1:	Correct exact answer. See scheme.
Note:	Give M0 A0 for writing down the correct exact answer without any evidence of substituting $t=\sqrt{2}$ into $\sin x=\frac{2t}{1+t^2}$
Note:	For reference, $\sin x=\frac{2}{3}\sqrt{2}=0.9428...$
(b)(ii)	**Way 1, Way 2 and Way 3**
M1:	Uses a correct trigonometric identity (or correct trigonometric identities) to find a correct expression which connects only $\cos x$ (or $\cos^2 x$) and t
A1*:	Correct proof
(b)(ii)	**Way 4**
M1:	Uses $\sin x=\frac{o}{h}$ and a correct Pythagoras method to express the adjacent edge of a triangle in terms of t.
A1*	Correct proof
(c)	
M1:	Uses at least one of $\sin\theta=\frac{2t}{1+t^2}$ or $\cos\theta=\frac{1-t^2}{1+t^2}$ to express $7\sin\theta+9\cos\theta+3$ in terms of t only
M1:	Uses both correct formula $\sin\theta=\frac{2t}{1+t^2}$ and $\cos\theta=\frac{1-t^2}{1+t^2}$ in $7\sin\theta+9\cos\theta+3=0$, multiplies both sides by $1+t^2$, forms a 3TQ and uses a correct method (e.g. using the quadratic formula, completing the square or a calculator approach) for solving their 3TQ to give $t=...$
M1:	Uses both correct formula $\sin\theta=\frac{2t}{1+t^2}$ and $\cos\theta=\frac{1-t^2}{1+t^2}$ in $7\sin\theta+9\cos\theta+3=0$, adopts a correct *applied* strategy to find at least one value of θ within the range $0<\theta\le 360°$ (or in radians $0<\theta\le 2\pi$) such that either • $\theta=2\arctan(\text{their found } t)$, where their found $t>0$ • $\theta=2(180°+\arctan(\text{their found } t))$, where their found $t<0$ • $\theta=2(180°-\arctan\lvert\text{their found } t\rvert)$, where their found $t<0$
A1:	Correct answer only of $\theta=143.1°, 292.6°$
Note:	Give A0 for extra solutions given within the range $0<\theta\le 360°$
Note:	Ignore extra solutions outside the range $0<\theta\le 360°$ for the A mark
Note:	Give 3rd M0 for $\frac{\theta}{2}=\{71.565..., 146.309...\}$ without attempting to find θ
Note	Give 3rd M0 for $\frac{\theta}{2}=\{71.565..., 146.309...\}\Rightarrow\theta=\{35.782..., 73.154...\}$
Note:	In degrees, $\frac{\theta}{2}=\{71.565..., 251.565..., -33.690..., 146.309...\}$
Note:	Working in radians gives $\frac{\theta}{2}=\{1.249..., 2.553...\}\Rightarrow\theta=\{2.498..., 5.107...\}$

Q2.

Question	Scheme	Marks	AOs
(i)	$\text{lhs} = \cot x + \tan\left(\frac{x}{2}\right) = \frac{1-t^2}{2t} + t$	M1	1.1a
	$\frac{1-t^2}{2t} + t = \frac{1+t^2}{2t}\left(= \frac{1}{\sin x}\right) = \operatorname{cosec} x$*	A1*	2.1
		(2)	
(ii)(a)	$x = 0 \Rightarrow H = 90 - 30\cos(0) - 40\sin(0) = 90 - 30 = 60$	B1	1.1b
		(1)	
(b)	$H = 90 - 30\cos 120x - 40\sin 120x = 90 - 30\left(\frac{1-t^2}{1+t^2}\right) - 40\left(\frac{2t}{1+t^2}\right)$	M1	1.1b
	$= \frac{90 + 90t^2 - 30 + 30t^2 - 80t}{1+t^2}$	M1	1.1b
	$= \frac{120t^2 - 80t + 60}{1+t^2}$*	A1*	2.1
		(3)	
(c)	$\frac{120t^2 - 80t + 60}{1+t^2} = 100 \Rightarrow 120t^2 - 80t + 60 = 100 + 100t^2$	M1	3.4
	$20t^2 - 80t - 40 = 0$	A1	1.1b
	$t = \frac{4 \pm \sqrt{16+8}}{2} \Rightarrow 60x = \tan^{-1}\left(2+\sqrt{6}\right)$ or $60x = \tan^{-1}\left(2-\sqrt{6}\right)$	M1	3.4
	$60x = \tan^{-1}\left(2+\sqrt{6}\right) = 77.33... \Rightarrow x = ...$	dM1	3.1b
	$x = 1.29$	A1	3.2a
		(5)	
			(11 marks)

Notes
(ii) M1: Selects the correct expression for $\cot x$ in terms of t and substitutes this and t into the lhs A1*: Fully correct proof. Allow correct work leading to $\frac{1+t^2}{2t} = \operatorname{cosec} x$ (ii)(a) B1: Demonstrates that when $x = 0$, $H = 60$ (b) M1: Uses the correct formulae to obtain H in terms of t M1: Correct method to obtain a common denominator A1*: Collects terms and simplifies to obtain the printed answer with no errors (c) M1: Uses $H = 100$ with the model and multiplies up to obtain a quadratic equation in t A1: Correct 3TQ M1: Solves their 3TQ in t and proceeds to obtain values of $60x$ as suggested by the model M1: A fully correct strategy to identify the required value of x from the positive root of the quadratic equation in t A1: awrt 1.29 Attempts in radians can score all but the final mark in (c). (Gives $60x = 1.3...$ etc.)

Q3.

Question	Scheme	Marks	AOs
(a)	$\sec x - \tan x = \dfrac{1}{\dfrac{1-t^2}{1+t^2}} - \dfrac{2t}{1-t^2}$	M1	2.1
	$= \dfrac{1+t^2}{1-t^2} - \dfrac{2t}{1-t^2} = \dfrac{1-2t+t^2}{1-t^2}$	M1	1.1b
	$= \dfrac{(1-t)^2}{(1-t)(1+t)} = \dfrac{1-t}{1+t}$ *	A1*	2.1
		(3)	
(b)	$\dfrac{1-\sin x}{1+\sin x} = \dfrac{1-\dfrac{2t}{1+t^2}}{1+\dfrac{2t}{1+t^2}}$	M1	1.1a
	$= \dfrac{1+t^2-2t}{1+t^2+2t}$	M1	1.1b
	$= \dfrac{(1-t)^2}{(1+t)^2} = \left(\dfrac{1-t}{1+t}\right)^2 = (\sec x - \tan x)^2$ *	A1*	2.1
		(3)	
			(6 marks)

Notes

(a)

M1 Uses $\sec x = \dfrac{1}{\cos x}$ and the t-substitutions for both $\cos x$ and $\tan x$ to obtain an expression in terms of t.

M1 Sorts out the $\sec x$ term, and puts over a common denominator of $1-t^2$

A1* Factorises both numerator and denominator (must be seen) and cancels the $(1+t)$ term to achieve the answer.

(b)

M1 Uses the t-substitution for $\sin x$ in both numerator and denominator.

M1 Multiples through by $1+t^2$ in numerator and denominator.

A1* Factorises both numerator and denominator and makes the connection with part (a) to achieve the given result.

Q4.

Question	Scheme	Marks	AOs
	$\mathrm{h}(x) = 45 + 15\sin x + 21\sin\left(\frac{x}{2}\right) + 25\cos\left(\frac{x}{2}\right)$		
(a)	$\frac{\mathrm{dh}}{\mathrm{d}x} = 15\cos x + \frac{21}{2}\cos\left(\frac{x}{2}\right) - \frac{25}{2}\sin\left(\frac{x}{2}\right)$	M1	1.1b
	$\frac{\mathrm{dh}}{\mathrm{d}x} = ... + ..\frac{1-t^2}{1+t^2} - ..\frac{2t}{1+t^2}$	M1	1.1a
	E.g. $\frac{\mathrm{dh}}{\mathrm{d}x} = ..\left(2\left(\frac{1-t^2}{1+t^2}\right)^2 - 1\right) + ...$ or $\frac{\mathrm{dh}}{\mathrm{d}x} = ..\frac{1-\left(\frac{2t}{1-t^2}\right)^2}{1+\left(\frac{2t}{1-t^2}\right)^2} + ...$	M1	3.1a
	E.g. $\frac{dh}{\mathrm{d}x} = 15\left(2\left(\frac{1-t^2}{1+t^2}\right)^2 - 1\right) + \frac{21}{2}\left(\frac{1-t^2}{1+t^2}\right) - \frac{25}{2}\left(\frac{2t}{1+t^2}\right)$	A1	1.1b
	$... = \frac{15[4(1-t^2)^2 - 2(1+t^2)^2] + 21(1-t^2)(1+t^2) - 50t(1+t^2)}{2(1+t^2)^2}$	M1	2.1
	$... = \frac{9t^4 - 50t^3 - 180t^2 - 50t + 51}{2(1+t^2)^2} = \frac{(t^2 - 6t - 17)(9t^2 + 4t - 3)}{2(1+t^2)^2}$ *	A1*	2.1
		(6)	

Question	Scheme	Marks	AOs
(a) ALT 1	$\mathrm{h}(x) = ... + 21\left(\frac{2t}{1+t^2}\right) + 25\left(\frac{1-t^2}{1+t^2}\right)$	M1	1.1a
	$= ... + 15\left[2\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)\right] + ...$ or $= ... + 15\left(\frac{2\left(\frac{2t}{1-t^2}\right)}{1+\left(\frac{2t}{1-t^2}\right)^2}\right) + ...$	M1	2.1
	$\mathrm{h}(x) = 45 + \frac{15(4t(1-t^2)) + 42t(1+t^2) + 25(1-t^4)}{(1+t^2)^2}$	M1	1.1b
	$\mathrm{h}(x) = 45 - \frac{25t^4 + 18t^3 - 102t - 25}{(1+t^2)^2}$ or $\frac{20t^4 - 18t^3 + 90t^2 + 102t + 70}{(1+t^2)^2}$	A1	1.1b
	$\frac{\mathrm{dh}}{\mathrm{d}x} = \frac{\mathrm{dh}}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{('u')(1+t^2)^2 - ('u')(4t(1+t^2))}{(1+t^2)^4} \times \frac{1}{4}(1+t^2)$	M1	3.1a
	$... = \frac{9t^4 - 50t^3 - 180t^2 - 50t + 51}{2(1+t^2)^2} = \frac{(t^2 - 6t - 17)(9t^2 + 4t - 3)}{2(1+t^2)^2}$ *	A1*	2.1
		(6)	
(b)(i)	Accept any value between $\frac{1}{40} = 0.025$ and $\frac{1}{60} \approx 0.167$ inclusive.	B1	3.3
(ii)	Suitable for times since the graphs both oscillate bi-modally with about the same periodicity.	B1	3.4
	Not suitable for predicting heights since the heights of the peaks vary over time, but the graph of h (x) has fixed peak height.	B1	3.5b
		(3)	

Question	Scheme	Marks	AOs
(c)	Solves at least one of the quadratics $t=\dfrac{6\pm\sqrt{36-4\times1\times17}}{2}=3\pm\sqrt{26}$ or $t=\dfrac{-4\pm\sqrt{16-4\times9\times(-3)}}{18}=\dfrac{-2\pm\sqrt{31}}{9}$	M1	1.1b
	Finds corresponding x values, $x=4\tan^{-1}(t)$ for at least one value of t from the $9t^2+4t-3$ factor.	M1	1.1b
	One correct value for these x. e.g. $x=$ awrt -2.797 or $9.770, 1.510$	A1	1.1b
	Maximum peak height occurs at smallest positive value of x, from first graph, but the third of these peaks needed, so $t=1.509...+8\pi=26.642...$ is the required time.	M1	3.4
	$x=26.642....$ corresponds to 26 hours and 39 minutes (nearest minute) after 08:00 on 3rd January. (Allow if a different greatest peak height used.)	M1	3.4
	Time of greatest tide height is approximately 10:39 (am) (also allow 10:38 or 10:40)	A1	3.2a
		(6)	
			(15 marks)

Notes

(a)

M1	Differentiates $\mathrm{h}(x)$.
M1	Applies t-substitution to both $\left(\dfrac{x}{2}\right)$ terms with their coefficients.
M1	Forms a correct expression in t for the $\cos x$ term, using double angle formula and t-substitution, or double 't'-substitution.
A1	Fully correct expression in t for $\dfrac{\mathrm{dh}}{\mathrm{d}x}$.
M1	Gets all terms over the correct common factor. Numerators must be appropriate for their terms.
A1*	Achieves the correct answer via expression with correct quartic numerator before factorisation.

Notes Continued

(a)

ALT 1

M1	Applies t-substitution to both $\left(\dfrac{x}{2}\right)$ terms.
M1	Forms a correct expression in t for the $\sin x$ term, using double angle formula and t-substitution, or double 't'-substitution.
M1	Gets all terms in t over the correct common factor. Numerators must be appropriate for their terms. May include the constant term too.
A1	Fully correct expression in t for $\mathrm{h}(x)$.
M1	Differentiates, using **both** chain rule and quotient rule with their 'u'.
A1*	Achieves the correct answer via expression with correct quartic numerator before factorisation.
NOTE	The individual terms may be differentiated before putting over a common denominator. In this case score the third M for differentiating with chain rule and quotient rule, then return to the original scheme.

(b) (i)	
B1	Any value between $\frac{1}{40}$ (e.g. taking $\mathrm{h}(0)$ as reference point) or $\frac{1}{60}$ (taking lower peaks as reference). NB: Taking high peak as reference gives $\frac{1}{50}$.
(b)(ii)	
B1	Should mention both the bimodal nature and periodicity for the actual data match the graph of h.
B1	Mentions that the heights of peaks vary in each oscillation.
(c)	
M1	Solves (at least) one of the quadratic equations in the numerator.
M1	Must be attempting to solve the quadratic factor from which the solution comes $(9t^2+4t-3)$ and using $t=\tan\left(\frac{x}{4}\right)$ to find a corresponding value for x.
A1	At least one correct x value from solving the requisite quadratic: awrt any of -2.797, 1.510, 9.770, 14.076, 22.336, 26.642, 34.902 or 39.208.
M1	Uses graph of h to pick out their $x = 26.642$ as the time corresponding to the third of the higher peaks, which is the highest of the peaks on 4th January on the tide height graph. As per scheme or allow if all times listed and correct one picked.
M1	Finds the time for one of the values of t corresponding to the highest peaks. E.g. 1.5096...~ 09:31 (3rd January) or 14.076... ~ 22:05 (3rd January) or 26.642…~ 10:39 (4th January) or 39.208…~ 23:13 (4th January). (Only follow through on use of the smallest positive t solution $+\ 4k\pi$.)
A1	Time of greatest tide height on 4th January is approximately 10:39. Also allow 10:38 or 10:40.