Oxford Cambridge and RSA

Tuesday 25 June 2019 – Morning

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)
- Computer with appropriate software

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

COMPUTING RESOURCES
- Candidates will require access to a computer with a computer algebra system, a spreadsheet, a programming language and graph-plotting software throughout the examination.

INFORMATION
- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **8** pages.

Answer **all** the questions.

1 A family of curves is given by the parametric equations

$$x(t) = \cos(t) - \frac{\cos((m+1)t)}{m+1} \quad \text{and} \quad y(t) = \sin(t) - \frac{\sin((m+1)t)}{m+1}$$

where $0 \leqslant t < 2\pi$ and m is a positive integer.

(a) **(i)** Sketch the curves in the cases $m = 3$, $m = 4$ and $m = 5$ on separate axes in the Printed Answer Booklet. **[3]**

(ii) State one common feature of these three curves. **[1]**

(iii) State a feature for the case $m = 4$ which is absent in the cases $m = 3$ and $m = 5$. **[1]**

(b) **(i)** Determine, in terms of m, the values of t for which $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ but $\frac{\mathrm{d}y}{\mathrm{d}t} \neq 0$. **[4]**

(ii) Describe the tangent to the curve at the points corresponding to such values of t. **[1]**

(c) **(i)** Show that the curve lies between the circle centred at the origin with radius

$$1 - \frac{1}{m+1}$$

and the circle centred at the origin with radius

$$1 + \frac{1}{m+1}.$$ **[2]**

(ii) Hence, or otherwise, show that the area A bounded by the curve satisfies

$$\frac{m^2\pi}{(m+1)^2} < A < \frac{(m+2)^2\pi}{(m+1)^2}.$$ **[1]**

(iii) Find the limit of the area bounded by the curve as m tends to infinity. **[1]**

(d) The arc length of a curve defined by parametric equations $x(t)$ and $y(t)$ between points corresponding to $t = c$ and $t = d$, where $c < d$, is

$$\int_c^d \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Use this to show that the length of the curve is independent of m. **[6]**

2 **(a)** Prove that if x and y are integers which satisfy $x^2 - 2y^2 = 1$, then x is odd and y is even. **[3]**

(b) Create a program to find, for a fixed positive integer s, all the positive integer solutions (x, y) to the equation $x^2 - 2y^2 = 1$ where $x \leqslant s$ and $y \leqslant s$. Write out your program in the Printed Answer Booklet. **[3]**

(c) Use your program to find all the positive integer solutions (x, y) to the equation $x^2 - 2y^2 = 1$ where $x \leqslant 600$ and $y \leqslant 600$. Give the solutions in ascending order of the value of x. **[1]**

(d) By writing the equation $x^2 - 2y^2 = 1$ in the form $(x + \sqrt{2}y)(x - \sqrt{2}y) = 1$ show how the first solution (the one with the lowest value of x) in your answer to part **(c)** can be used to generate the other solutions you found in part **(c)**. **[4]**

(e) What can you deduce about the number of positive integer solutions (x, y) to the equation $x^2 - 2y^2 = 1$? **[1]**

In the remainder of this question T_m is the m^{th} triangular number, the sum of the first m positive integers, so that $T_m = \dfrac{m(m+1)}{2}$.

(f) Create a program to find, for a fixed positive integer t, all pairs of positive integers m and n which satisfy $T_m = n^2$ where $m \leqslant t$ and $n \leqslant t$. Write out your program in the Printed Answer Booklet. **[2]**

(g) Use your program to find all pairs of positive integers m and n which satisfy $T_m = n^2$ where $m \leqslant 300$ and $n \leqslant 300$. Give the pairs in ascending order of the value of m. **[1]**

(h) By comparing your answers to part **(c)** and part **(g)**, or otherwise, prove that there are infinitely many triangular numbers which are perfect squares. **[5]**

3 This question concerns the family of differential equations

$$\frac{dy}{dx} = 1 - x^a y \ (*)$$

where a is –1, 0 or 1.

(a) Determine and describe geometrically the isoclines of (*) when

(i) $a = -1$, **[2]**

(ii) $a = 0$, **[2]**

(iii) $a = 1$. **[2]**

(b) In this part of the question $a = 0$.

(i) Write down the solution to (*) which passes through the point (0, b) where $b \neq 1$. **[1]**

(ii) Write down the equation of the asymptote to this solution. **[1]**

(c) In this part of the question $a = -1$.

(i) Write down the solution to (*) which passes through the point (c, d) where $c \neq 0$. **[1]**

(ii) Describe the relationship between c and d when the solution in part **(i)** has a stationary point. **[4]**

(d) In this part of the question $a = 1$.

(i) The standard Runge-Kutta method of order 4 for the solution of the differential equation $\mathrm{f}(x, y) = \frac{\mathrm{d}y}{\mathrm{d}x}$ is as follows.

$$k_1 = h\mathrm{f}(x_n, y_n)$$

$$k_2 = h\mathrm{f}\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\mathrm{f}\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h\mathrm{f}(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Construct a spreadsheet to solve (*) in the case $x_0 = 0$ and $y_0 = 1.5$. State the formulae you have used in your spreadsheet. **[4]**

(ii) Use your spreadsheet with $h = 0.05$ to find an approximation to the value of y when $x = 1$. **[1]**

(iii) The solution to (*) in which $x_0 = 0$ and $y_0 = 1.5$ has a maximum point (r, s) with $0 < r < 1$. Use your spreadsheet with suitable values of h to estimate r to two decimal places. Justify your answer. **[2]**

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