Oxford Cambridge and RSA

Friday 23 October 2020 – Afternoon

A Level Further Mathematics A

Y545/01 Additional Pure Mathematics

Time allowed: 1 hour 30 minutes

8319807696

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for A Level Further Mathematics A
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **4** pages.

ADVICE
- Read each question carefully before you start your answer.

Answer **all** the questions.

1 The following Cayley table is for a set $\{a, b, c, d\}$ under a suitable binary operation.

	a	b	c	d
a	b		a	
b				
c			c	
d	d			a

(a) Given that the Latin square property holds for this Cayley table, complete it using the table supplied in the Printed Answer Booklet. **[4]**

(b) Using your completed Cayley table, explain why the set does **not** form a group under the binary operation. **[1]**

2 For $x, y \in \mathbb{R}$, the function f is given by $f(x, y) = 2x^2y^7 + 3x^5y^4 - 5x^8y$.

(a) Prove that $x\mathrm{f}_x + y\mathrm{f}_y = n\mathrm{f}$, where n is a positive integer to be determined. **[5]**

(b) Show that $x\mathrm{f}_{xx} + y\mathrm{f}_{xy} = (n-1)\mathrm{f}_x$. **[4]**

3 For integers $n \geqslant 0$, $I_n = \int_0^1 \frac{x^n}{1+x^2}\,\mathrm{d}x$.

(a) For integers $n \geqslant 2$, show that $I_n + I_{n-2} = \frac{1}{n-1}$. **[3]**

(b) **(i)** Determine the exact value of I_{10}. **[4]**

(ii) Deduce that $\pi < 3\frac{107}{315}$. **[2]**

4 Points A, B and C have position vectors **a**, **b** and **c** respectively, relative to origin O.

It is given that $\mathbf{b} \times \mathbf{c} = \mathbf{a}$ and that $|\mathbf{a}| = 3$.

(a) Determine each of the following as either a single vector or a scalar quantity.

(i) $\mathbf{c} \times \mathbf{b}$ **[1]**

(ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ **[2]**

(iii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ **[2]**

(b) Describe a geometrical relationship between the points O, A, B and C which can be deduced from

(i) the statement $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, **[1]**

(ii) the result of **(a)(iii)**. **[1]**

5 A designer intends to manufacture a product using a 3-D printer. The product will take the form of a surface S which must meet a number of design specifications. The designer chooses to model S with the equation $z = y\cosh x$ for $-\ln 20 \leqslant x \leqslant \ln 20$, $-2 \leqslant y \leqslant 2$.

(a) **(i)** In the Printed Answer Booklet, on the axes provided, sketch the section of S given by $y = 1$. **[1]**

(ii) One of the design specifications of the product is that this section should have a length no greater than 20 units.

Determine whether the product meets this requirement according to the model. **[4]**

(b) **(i)** In the Printed Answer Booklet, on the axes provided, sketch the contour of S given by $z = 1$. **[1]**

(ii) When this contour is rotated through 2π radians about the x-axis, the surface T is generated. The surface area of T is denoted by A.

Show that A can be written in the form $k\pi\int_0^{\ln 20} \frac{1}{\cosh^3 x}\sqrt{\cosh^4 x + \cosh^2 x - 1}\,\mathrm{d}x$ for some integer k to be determined. **[5]**

(iii) A second design specification is that the surface area of T must not be greater than 20 square units.

Use your calculator to decide whether the product meets this requirement according to the model. **[2]**

6 The group G consists of the set {3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36} under $\times_{39}$, the operation of multiplication modulo 39.

(a) List the possible orders of proper subgroups of G, justifying your answer. **[2]**

(b) List the elements of the subset of G generated by the element 3. **[1]**

(c) State the identity element of G. **[1]**

(d) Determine the order of the element 18. **[2]**

(e) Find the two elements g_1 and g_2 in G which satisfy $g \times_{39} g = 3$. **[3]**

The group H consists of the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} under $\times_{13}$, the operation of multiplication modulo 13. You are given that G is isomorphic to H.

A student states that G is isomorphic to H because each element $3x$ in G maps directly to the element x in H (for x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12).

(f) Explain why this student is incorrect. **[1]**

7 Throughout this question, n is a positive integer.

(a) Explain why $n^5 \equiv n \pmod 5$. **[1]**

(b) By proving that $n^5 \equiv n \pmod 2$, show that $n^5 \equiv n \pmod{10}$. **[3]**

(c) **(i)** Prove that $n^5 - n$ is divisible by 30 for all positive integers n. **[5]**

(ii) Is there an integer N, greater than 30, such that $n^5 - n$ is divisible by N for all positive integers n? Justify your answer. **[1]**

8 The sequence $\{u_n\}$ of positive real numbers is defined by $u_1 = 1$ and $u_{n+1} = \dfrac{2u_n + 3}{u_n + 2}$ for $n \geqslant 1$.

(a) Prove by induction that $u_n^2 - 3 < 0$ for all positive integers n. **[6]**

(b) By considering $u_{n+1} - u_n$, use the result of part **(a)** to show that $u_{n+1} > u_n$ for all positive integers n. **[3]**

The sequence $\{u_n\}$ has a limit for $n \to \infty$.

(c) Find the limit of the sequence $\{u_n\}$ as $n \to \infty$. **[2]**

(d) Describe as fully as possible the behaviour of the sequence $\{u_n\}$. **[1]**

END OF QUESTION PAPER