

## Gravitational fields

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### Gravitational fields

A force field is a region in which a body experiences a non-contact force. A force field can be represented as a vector, the direction of which must be determined by inspection.

- Gravity is a universal attractive force which acts between all matter
  - magnitude of a force between point masses,  $F = \frac{Gm_1m_2}{r^2}$  where G is the gravitational constant
- A gravitational field can be represented by field lines - also known as lines of force.
  - This is the path followed by a small mass placed close to a massive body.
  - Note that for a radial field, the field lines point towards the centre. In a uniform field e.g. close to the Earth's surface, field lines act straight down - parallel to each other and evenly spaced.
- The gravitational field strength, g, is the force per unit mass on a small test mass placed in the field.

$$g = \frac{F}{m}$$

- In a radial field, the magnitude of:

$$g = \frac{GM}{r^2}$$

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### Gravitational potential

- **Gravitational potential** at a point is the gravitational potential energy per unit mass of a small test mass.
  - This is equal to the work done per unit mass to move an object from infinity (where potential = 0) to that point.

$$\text{gravitational potential, } V = \frac{W}{m} \text{ unit: J kg}^{-1}$$

$$\text{work done moving mass } m: \Delta W = m\Delta V$$

$$\text{gravitational potential in a radial field: } V = -\frac{GM}{r}$$

- The negative sign is due to the reference point being infinity, and the fact that other than at infinity the force is in fact attractive.

- $\Delta V$  can be found from the area of a g-r graph
- **Equipotentials** are surfaces of constant potential - no work needs to be done to move along an equipotential surface.
- **Potential gradient** at a point in a gravitational field is the change of potential per metre at that point
- In general, for  $\Delta V$  over a small distance  $\Delta r$ , potential gradient =  $\frac{\Delta V}{\Delta r}$
- Gravitational field strength is the negative of potential gradient:

$$g = -\frac{\Delta V}{\Delta r}$$

## Orbits and satellites

If an object is moving parallel to a planet's surface at the correct speed such that the centripetal force required is matched exactly by the force of gravity, it will orbit.

For a satellite orbiting at distance  $r$  from the centre of a planet:  $\frac{GM_{\text{planet}}m}{r^2} = \frac{mv^2}{r} = mr\omega^2$   
 showing  $m$  irrelevant

- For **geostationary** orbit,  $T_{\text{sat}} = T_{\text{planet}}$ , so for earth  $T \approx 86\,400$  s

$$-T = \frac{2\pi}{\omega}$$

Kepler's 3rd Law proof and derivation:

- For an object in orbit around mass  $M$ :

$$1. \quad \frac{GM}{r^2} = r\omega^2 \text{ so } \frac{GM}{r^3} = \omega^2$$

$$2. \quad \text{Combining with } T = \frac{2\pi}{\omega} \text{ gives } \frac{GM}{r^3} = \frac{4\pi^2}{T^2}, \text{ or } T^2 = \frac{4\pi^2}{GM}r^3$$

3. Everything is constant except  $T$  and  $r$ , meaning  $T^2 \propto r^3$  - Kepler's 3rd Law

$$4. \quad \text{To further prove K3L, if } T^2 = \frac{4\pi^2}{GM}r^3, \text{ taking logarithms gives } \log(T^2) = \log\left(\frac{4\pi^2}{GM}r^3\right)$$

$$5. \quad \log(T^2) = \log\left(\frac{4\pi^2}{GM}\right) + \log(r^3)$$

$$6. \quad 2\log(T^2) = 3\log(r) + \log\left(\frac{4\pi^2}{GM}\right)$$

$$7. \quad \log(T^2) = 1.5\log(r) + 0.5\log\left(\frac{4\pi^2}{GM}\right)$$

$$8. \quad \log(T^2) = 1.5\log(r) + \log\left(\sqrt{\frac{4\pi^2}{GM}}\right)$$

9. Hence a graph of  $\log T$  against  $\log r$  has gradient 1.5 and positive y-intercept of  $\frac{2\pi}{\sqrt{GM}}$

## Escape velocity

For an object to go into orbit once launched rather than fall back to Earth, it must never run out of kinetic energy. So supplied  $\frac{E_k}{m} \geq v$

Equating  $E_k$  and  $V \cdot m$  allows us to work out that:

$$\text{escape velocity, } v = \sqrt{\frac{2GM}{r}}$$

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## Orbits and satellites

A satellite  $E_k = \frac{1}{2}mv^2$ . Equating forces in orbit gives  $\frac{mv^2}{r} = \frac{GMm}{r^2}$  or  $v^2 = \frac{GM}{r}$

Hence to be in orbit,  $E_k = \frac{GMm}{2r}$

Potential energy is calculated from gravitational potential:  $E_p = -\frac{GM}{r} \cdot m$

The total energy is the sum:  $E_T = \frac{GMm}{2r} + \left(-\frac{GMm}{r}\right)$

$$E_T = -\frac{GMm}{r}$$